Better Error Estimation for the mixed Szász Baskakov type operators

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Abstract: This paper introduced a generalization of mixed summation integral type operators with Szász and Baskakov Basis, so-called Szász Baskakov operator. Then the moment estimates of these operators have been obtained and the uniform convergence has been established. Further, the quantitative approximation and local approximation behavior of the operators has been studied using modulus of continuity and Lipschitz class function. Then, it has been proved that the rate of convergence of the proposed operators is better than their primitives. In the last section, r−th order generalization of modified operators has been introduced and their rate of convergence has been estimated.

1. Introduction

Better error estimation of a linear positive operator is one of most important problem in approximation theory [1-5], which allows us much faster rate of convergence to the function being approximated [6-7]. Recently, some relating investigations were accomplished for Bernstein polynomials [8-9], Szász-Mirakjan operators [4], Meyer-Konig and Zeller operators [10], Bernstein Chlodovsky operators [1] and Szász-Mirakjan-Beta operator [3]. In this paper, investigate better error estimation for mixed Szász and Baskakov operator.
In 2004, Gupta and Gupta [6] introduced the operator as

$$S_n(f;x) = (n-1)\sum_{v=1}^{\infty} s_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)f(t)dt + e^{-nx}f(0), x \in [0,\infty). \quad (1.1)$$

where $s_{n,v}(x) = e^{-nx}\frac{(nx)^v}{v!}$

and $b_{n,v}(t) = \binom{n+v-1}{v} t^v(1+t)^{-n-v}$.

are respectively Szász and Baskakov basis functions. Moreover

$$S_n(f;x) = \int_0^{\infty} W_n(x,t)f(t)dt$$

where

$$W_n(x,t) = (n-1)\sum_{v=1}^{\infty} s_{n,v}(x) b_{n,v-1}(t)f(t) + e^{-nx}\delta(t).$$

$\delta(t)$ being Dirac delta function. It can be easily observed that operators are linear positive operator and reproduce constant functions. The behaviour of these operators is much similar to operators introduced by Gupta and Srivastava in 1995 [12]. The basic difference between two operators is that operators (1.1) are discretely defined at point zero. Also, approximation properties of these two operators are different.
Recently, [7] Gupta and Erkus, obtain pointwise rate of convergence, asymptotic formula and error estimation in simultaneous approximation for the operator (1.1). In 2006, Sinha and Singh [11] estimates the rate of convergence for the operator (1.1) with functions having derivatives of bonded variation.
Lemma 1 [12] Let the function $\mu_{n,m}(x)$, $m \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$, be defined as

$$\mu_{n,m}(x) = (n-1)\sum_{v=1}^{\infty} s_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)(t-x)^m dt + e^{-nx}(-x)^m,$$

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then

$$\mu_{n,0}(x) = 1, \mu_{n,1}(x) = \frac{2x}{n-2}, \mu_{n,2}(x) = \frac{nx(x+2)+6x^2}{(n-2)(n-3)}$$

also the recurrence relation:

$$(n-m-2)\mu_{n,m+1}(x) = x[\mu_{n,m}^{(1)}(x) + m(x+2)\mu_{n,m-1}(x)] + [m+2x(m+1)]\mu_{n,m}(x); n > m+2$$

consequently for each $x \in [0,\infty)$, this recurrence relation that

$$\mu_{n,m}(x) = O(n^{-[(m+1)/2]}).$$

Remark 1 It can be easily verified from lemma 1 that, for $x \in [0,\infty)$, $e_i = t^i$,

$$S_n(e_0;x) = 1,$$
$$S_n(e_1;x) = \frac{nx}{n-2},$$
$$S_n(e_2;x) = \frac{n^2x^2+2nx}{(n-2)(n-3)},$$

and in general, for $x \in [0,\infty)$, $n \in \mathbb{N}$ and $n \geq i$, $x \in [0,\infty)$,

$$S_n(e_i;x) = \frac{(n-i-2)!}{(n-2)!}(nx)^i + i(i-1)\frac{(n-i-2)!}{(n-2)!}(nx)^{i-1} + O(n^{-2}).$$

2. Construction of Operator

This section, modify the operator for better error estimation such that linear function are preserved. First consider the Banach lattice

$$C_\gamma[0,\infty) = \{f \in C[0,\infty): |f(x)| \leq M(1+x)^\gamma forsomeM > 0, \gamma > 0\}$$

endowed with norm

$$\| f \|_\gamma = \sup_{x\in[0,\infty)} \frac{|f(x)|}{(1+x)^\gamma}.$$

Clearly the set $\{e_0, e_1, e_2\}$ is a K_+-subset of $C_\gamma[0,\infty)$ and the space $C_\gamma[0,\infty)$ is isomorphic to $C[0,1]$ (see [5]).
Define sequence $\{r_n(x)\}$ of real valued continues functions on $[0,\infty)$ with $0 \leq r_n(x) < \infty$, as

$$r_n(x) = \frac{(n-2)x}{n}.$$

Replace x in definition of operator (1.1) with $r_n(x)$. Therefore modified operator is given as

$$S_n^*(f;x) = (n-1)\sum_{v=1}^{\infty} s_{n,v}^*(x)\int_0^{\infty} b_{n,v-1}(t)f(t)dt + e^{-nr_n(x)}f(0), x \in [0,\infty), \quad (1)$$

where

$$s^*_{n,v}(x) = e^{-nr_n(x)}\frac{(nr_n(x))^v}{v!}$$

and $x \in [0,\infty)$, $n \in \mathbb{N}$, the term $b_{n,v}(t)$ is baskakov basis. Moreover

$$S_n(f;x) = \int_0^\infty W^*_n(x,t)f(t)dt$$

where

$$W^*_n(x,t) = (n-1)\sum_{v=1}^{\infty} s^*_{n,v}(x)b_{n,v-1}(t)f(t) + e^{-nr_n(x)}\delta(t).$$

Lemma 2 For $x \in [0,\infty)$ and $n \in \mathbb{N}$ and $N > 3$,

$$S^*_n(e_0;x) = 1,$$
$$S^*_n(e_1;x) = x,$$
$$S^*_n(e_2;x) = \frac{(n-2)x^2+2x}{(n-3)}.$$

Lemma 3 For $x \in [0,\infty)$ and $n \in \mathbb{N}$, $n > 3$ and with $\phi_x = t - x$,

$$S^*_n(\phi_x;x) = 0,$$
$$S^*_n(\phi_x^2;x) = \frac{x^2+2x}{n-3}.$$

By lemma 2, for $h(t) = at + b$, where a, b any real constants, $S^*_n(h(t);x) = h(x)$ that is, operator $S^*_n(f;x)$ preserves the linear function.

For fix $c > 0$, define the lattice homomorphism $H_c: C[0,\infty) \to C[0,c]$ defined by $H_c(f) = f|_{[0,c]}$ for every $f \in C[0,\infty)$. Therefore, observe that, for each $i = 0,1,2,$

$$\lim_{n\to\infty} H_c(S^*_n(e_i)) = H_c(e_i) \quad (2)$$

uniformly on $[0,c]$.

Theorem A (Theorem 4.1.4(vi) of [5]) Let X be a compact set and H be a confinal subspace of $C(X)$. If E is a banach lattice, $S: C(X) \to E$ is a lattice homomorphism and if $\{L_n\}$ is a sequence of positive linear operators from $C(X)$ in E such that $lim_{n\to\infty} L_n(h) = S(h)$ for all $h \in H$, then $lim_{n\to\infty} L_n(f) = f$ provided that f belongs to Korovkin closure of H.

Therefore by equation (2.2) and theorem A, directly get following result:

Theorem 1 For $x \in [0,c]$, $c > 0$, $S^*_n(f;x)$ converges uniformly on $f(x)$ as $n \to \infty$, provided $f \in C_\gamma[0,\infty)$, $\gamma > 0$.

3. Local Approximation Theorem

Let $C_B[0,\infty)$ be the space of all real valued continuous bounded function f on $[0,\infty)$, with supreme norm defined as $\| f \| = \sup_{x\in[0,\infty)} |f(x)|$.

The K −functional is defined as

$$K(f,\delta) = \inf_{g\in W^2_\infty}\{\| f - g \| + \delta \| g \|_{C^2[0,a]}\}, \quad (3)$$

where

$$W^2_\infty = \{f \in C_B[0,\infty): f', f'' \in C_B[0,\infty)\}. \quad (4)$$

By [2], there exist a constant $C > 0$ such that

$$K(f,\delta) \le C\omega_2(f,\sqrt{\delta}), \delta > 0, \quad (5)$$

and second order modulus of continuity is given as

$$\omega_2(f,\delta) = \sup_{0\le h\le\delta; x\in[0,\infty)} [f(x+2h) - 2f(x+h) + f(x)]. \quad (6)$$

Theorem 2 Let $f \in C_B[0,\infty)$, then for every $x \in [0,\infty)$ and for $n > 3$, $C > 0$,

$$|S^*_n(f;x) - f(x)| \le C\omega_2(f,\delta_n), \quad (7)$$

where $\delta_n = \sqrt{\frac{x^2+2x}{n-3}}$.

Proof. Let $g \in W^2_\infty$, by Taylor series

$$g(t) - g(x) = g'(x)(t-x) + \int_x^t g''(s)(t-s)ds$$

therefore, by linearity and lemma 3,

$$|S^*_n(g;x) - g(x)| \le S^*_n(\phi_x,x) \| g' \| + \frac{\|g''\|}{2} S^*_n(\phi_x^2;x)$$

$$= \frac{x^2+2x}{2(n-3)} \| g'' \|$$

hence

$$|S^*_n(f;x) - f(x)| \le |S^*_n(f-g;x) - (f-g)(x)| + |S^*_n(g;x) - g(x)|$$
$$\le 2 \| f - g \| + \frac{x^2+2x}{2(n-3)} \| g'' \|$$

on letting $\delta_n = \sqrt{\frac{x^2+2x}{(n-3)}}$, taking infumum over $g \in W^2_\infty$ and by property (3.3), get desired results.

4. Better Error Estimation

In this section, compute rate of convergence for new improved operator and prove that our improved operator has faster rate of convergence.

Recall the concept of modulus of continuity, the modulus of continuity of $f(x) \in C[0,\infty)$, denoted by $\omega(f,\delta)$, is defined by

$$\omega(f,\delta) = \sup_{|x-y|\le\delta, x,y\in[0,\infty)} |f(x) - f(y)| \quad (8)$$

The modulus of continuity possesses following properties (see [9])

$$\omega(f,\lambda\delta) \le (1+\lambda)\omega(f,\delta) \quad (9)$$

Theorem 3 For every $f \in C[0,\infty)$, $x \in [0,\infty)$, $n \in \mathbb{N}$ and $n > 3$,

$$|S^*_n(f;x) - f(x)| \le 2\omega(f,\delta_{n,x}), \quad (10)$$

where

$$\delta_{n,x} = \sqrt{\frac{x^2+2x}{n-3}}. \quad (11)$$

Proof. Let $f \in C[0,\infty)$ and $x \in [0,\infty)$. By the linearity and monotonicity of $S^*_n(f;x)$, for every $\delta > 0$ and $n > 3$, that

$$|S^*_n(f;x) - f| \le \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{S^*_n(\phi_x^2;x)}\right)$$

So, letting $\delta_{n,x} = \sqrt{S^*_n(\phi_x^2;x)}$ and take $\delta = \delta_{n,x}$, the proof is complete.

Remark 2 For Szász-Baskakov operators (1.1), for every $f \in C[0,\infty)$, $x \in [0,\infty)$, $n \in \mathbb{N}$ and $n > 3$,

$$|S_n(f,x) - f(x)| \le 2\omega(f,\alpha_{n,x}), \quad (12)$$

where

$$\alpha_{n,x} = \sqrt{\frac{nx(x+2)+6x^2}{(n-2)(n-3)}}.$$

error estimation in Theorem 3 is better than that for Szász-Baskakov operators given by (4.5), provided $f \in C[0,\infty)$ and $n > 3$. Indeed, it is clear that for better error estimation, to just show that $\delta_{n,x} \le \alpha_{n,x}$, which is as follows:

$$\delta_{n,x} \le \alpha_{n,x}$$
$$\Leftrightarrow \frac{x^2+2x}{n-3} \le \frac{nx(x+2)+6x^2}{(n-2)(n-3)}$$
$$\Leftrightarrow 8x^2 + 4x \ge 0$$

which holds true for all $x \ge 0$. Hence our claim is true.

Now try to compute rate of convergence with the help of the elements of Lipschitz class $Lip_M(\alpha)$, $(0 < \alpha \le 1)$.

As usual, a function $f \in Lip_M(\alpha)$, ($M > 0$ and $0 < \alpha \le 1$), if the inequality

$$|f(t) - f(x)| \le M|t - x|^\alpha \quad (13)$$

holds for all $t, x \in [0,\infty)$.

Theorem 4 For every $f \in Lip_M(\alpha)$, $x \in [0,\infty)$, $n \in \mathbb{N}$ and $n > 3$,

$$|S^*_n(f;x) - f| \le M\{\frac{x^2+2x}{n-3}\}^{\alpha/2}. \quad (14)$$

Proof. Using lemma 3 and Hölder inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$,

$$|S^*_n(f;x) - f| \le S^*_n(|f(t) - f(x)|;x)$$
$$\le MS^*_n(|t-x|^\alpha;x)$$
$$\le MS^*_n(\phi_x^2;x)^{\alpha/2}$$
$$\le M\{\frac{x^2+2x}{n-3}\}^{\alpha/2}.$$

Hence proof is completed.

Remark 3 For Szász-Baskakov operators (1.1), easily obtain that, for every $f \in Lip_M(\alpha)$, ($M > 0$ and $0 < \alpha \leq 1$) and $x \geq 0$,

$$|S_n^*(f;x) - f| \leq M\{\frac{nx(x+2)+6x^2}{(n-2)(n-3)}\}^{\alpha/2}. \quad (15)$$

Therefore by argument as in Remark 2, prove that improved operator S_n^* by means of the element of Lipschitz class functional has better error estimation as compare to operator (1.1).

5. A Voronovskaya type theorem

In this section, prove a Voronovskaya type theorem for the operator $S_n^*(f;x)$.

Lemma 4 For $x \in [0,c]$, $c > 0$, $lim_{n\to\infty} n^2 S_n^*(\phi_x^4, x) = 43x^4 + 12x^3$ uniformly.

Proof. By lemma 2 and simple computations, can write $n^2 S_n^*(\phi_x^4, x) = \frac{(43n^3+12n^2)x^4+(12n^3+48n^2)x^3}{(n-3)(n-4)(n-5)}$.

On letting limit as $n \to \infty$, get the result.

Theorem 5 For every $f \in C_\gamma[0,\infty)$ such that $f', f'' \in C_\gamma[0,\infty)$, $n \in \mathbb{N}$ and $n > 3$, have

$$\lim_{n\to\infty} n\{S_n^*(f;x) - f\} = \frac{x(x+2)}{2} f''(x) \quad (16)$$

uniformly with respect to $x \in [0,c]$, $c > 0$.

Proof. For $f, f', f'' \in C_\gamma[0,\infty)$ and $x \geq 0$, define

$$\Psi(t,x) = \frac{f(t)-f(x)-(t-x)f'(x)-\frac{1}{2}(t-x)^2 f''(x)}{(t-x)^2} \quad (17)$$

for $t \neq x$ and $\Psi(x,x) = 0$. Then clearly $\Psi(t,x) \in C_\gamma[0,\infty)$. Hence using Taylor's theorem, get

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + (t-x)^2\Psi(t,x)$$

by lemma 3, can obtain

$$n\{S_n^*(f;x) - f\} = \frac{n}{2}\left(\frac{x^2+2x}{n-3}\right) f''(x) + nS_n^*(\phi_x^2(t)\Psi(t,x);x). \quad (18)$$

By applying Cauchy-Schwarz inequality for the second term of above expression, can conclude

$$n|S_n^*(\phi_x^2(t)\Psi(t,x);x)| \leq (n^2 S_n^*(\phi_x^4(t);x))^{1/2}(S_n^*(\Psi^2(t,x);x))^{1/2}. \quad (19)$$

Let $\eta(t,x) = \Psi^2(t,x)$, can observe that $\eta(x,x) = 0$ and $\eta(t,x) \in C_\gamma[0,\infty)$. Therefore by theorem 1,

$$\lim_{n\to\infty} |S_n^*(\Psi^2(t,x);x)| = \lim_{n\to\infty} |S_n^*(\eta(t,x);x)| = \eta(x,x) = 0 \quad (20)$$

uniformly for $x \in [0,c]$, $c > 0$. by equations (5.3) and (5.5), can write

$$n\{S_n^*(f;x) - f\} = \frac{n}{2}\left(\frac{x^2+2x}{n-3}\right) f''(x) \quad (21)$$

on taking limit as $n \to \infty$, in equation (5.6), get desired result.

6. r-th order generalization

First consider the space $C_\gamma^{(r)}[0,\infty), r = 0,1,2\cdots$, so

$$C_\gamma^{(r)}[0,\infty) = \{f \in C_\gamma[0,\infty): f^{(r)} \in C_\gamma[0,\infty)\}. \quad (21)$$

For $r = 0$, $C_\gamma^{(0)}[0,\infty)$ considers with $C_\gamma[0,\infty)$.

Now consider r^{th} order generalization of positive linear operator $S_{n,r}^*$ as

$$S_{n,r}^*(f;x) = (n-1)\sum_{v=1}^{\infty}\sum_{i=0}^{r} s_{n,v}^*(x)\int_0^\infty b_{n,v-1}(t)\frac{(t-x)^i}{i!}f^{(i)}(t)dt + e^{-nr_n(x)}f(0), x \in [0,\infty), \quad (22)$$

where $f \in C_\gamma^{(r)}[0,\infty), \gamma > 0, r = 0,1,2\cdots$. can easily observe that $S_{n,0}^*(f;x) = S_n^*(f;x)$.

The operator $S_{n,r}^*$ can be written as

$$S_{n,r}^*(f,x) = \int_0^\infty \sum_{i=0}^{r} W_n^*(x,t)\frac{(t-x)^i}{i!}f^{(i)}(t)dt. \quad (23)$$

Theorem 6 For all $f \in C_\gamma^{(r)}[0,\infty)$, $\gamma > 0$, such that $f^{(r)} \in Lip_M(\alpha)$ and for every $x \geq 0$,

$$|S_{n,r}^*(f;x) - f(x)| \leq \frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r)S_{n,r}^*(|t-x|^{\alpha+r};x), \quad (24)$$

where $r = 1,2,...$ and $B(\alpha,r)$ is the beta function.

Proof. By defination of operator (6.3) and lemma 2,

$$f(x) - S_{n,r}^*(f;x) = \int_0^\infty W_n^*(x,t)\{f(x) - \sum_{i=0}^{r}\frac{(t-x)^i}{i!}f^{(i)}(t)\}dt \quad (25)$$

Also by Taylor's formula,

$$f(x) - \sum_{i=0}^{r}\frac{(t-x)^i}{i!}f^{(i)}(t) = \frac{(x-t)^r}{(r-1)!}\int_0^1 (1-s)^r(f^{(r)}(t+s(x-t)) - f^{(r)}(t))ds \quad (26)$$

as $f^{(r)} \in Lip_M(\alpha)$, therefore

$$|f^{(r)}(t+s(x-t)) - f^{(r)}(t)| \leq Ms^\alpha|x-t|^\alpha \quad (27)$$

using equations (6.6), (6.7) and definition of beta function, can easily write

$$|f(x) - \sum_{i=0}^{r}\frac{(t-x)^i}{i!}f^{(i)}(t)| \leq M\frac{|t-x|^{\alpha+r}}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r) \quad (28)$$

Finally by using equations (6.5) and (6.6), get desired result.

Theorem 7 For $x \in [0,c]$, $c > 0$ and every $f \in C_\gamma^{(r)}[0,\infty)$, $\gamma > 0$, such that $f^{(r)} \in Lip_M(\alpha)$, $S_{n,r}^*(f,x)$ converges uniformly on $f(x)$ as $n \to \infty$.

Proof. Define $g(t) = |t-x|^\alpha$ and $x \in [0,c]$, $c > 0$, by theorem 1, can write

$$\lim_{n\to\infty} S_{n,r}^*(g,x) = g(x) = 0.$$

Finally results follows from Theorem 6.

7. Conclusions

This paper, introduced modification of mixed summation integral type operators with Szász and Baskakov Basis. Further, have proved that our operator has better rate of convergence. In last, have given r-th order gernalization of modified operator and study their rate of convergence.

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